

# Uncertainty rescued: Bohr's complementarity for composite systems

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## Abstract

Generalized uncertainty relations may depend not only on the commutator relation of two observables considered, but also on mutual correlations, in particular, on entanglement. The equivalence between the uncertainty relation and Bohr's complementarity thus holds in a much broader sense than anticipated.

*Key words:* uncertainty relation, complementarity, quantum entanglement

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## 1 Introduction

Heisenberg's uncertainty relation [1,2] and Bohr's principle of complementarity [3] are at the heart of quantum theory. The relation of these two principles has, however, created much debate: the former is often interpreted to imply that one cannot detect for a given quantum state two conjugate observables with unlimited precision. The latter may be understood to mean [7] that a state with minimum dispersion of one observable (i.e. preparation of a respective eigenstate) implies maximum dispersion of the other (i.e. any of its eigenstates

will be found with equal probability). Two conjugate variables like position  $\hat{x}$  and momentum  $\hat{p}_x$  are, at the same time, constrained by a typical uncertainty relation and complementarity in the above sense. Rather than considering these constraints as inherent properties of the quantum state, reference is often made to measurement disturbances (“random kicks”), as a quasi-classical explanation. This notion has a long history: In the famous Bohr-Einstein debates in 1920’s [4], Bohr invariably refuted Einstein’s *gedanken* experiments (e.g. the ‘recoiling slit’) by using the standard uncertainty relation to prove his complementarity. Thereafter similar conclusions have been postulated based on the momentum uncertainty in Heisenberg’s microscope [5] and Feynman’s electron-light scattering scheme [6], respectively.

In the 1990’s this topic was revived by a *welcher-Weg* (‘which-path’) measurement scheme in quantum optics, theoretically proposed by Scully, Englert, and Walther [7], and experimentally realized recently by Rempe and his collaborators [8] based on atomic beams and by Kwiat *et al.* [9] based on photon beams. These investigators claimed that the complementarity can be enforced without any uncertainty relation at work by exploiting quantum entanglement [10] between an atom (a photon) and a ‘which-path’ detector: this detector allows to record the path by using the atom’s different internal states (the photon’s different polarisation states, respectively). The complementarity would thus appear as a more general and more fundamental principle than the uncertainty rule [11,12]. However, this conclusion, ‘complementarity without uncertainty relation’, has been questioned by Steuernagel [13]. Based on a formal spin-1/2 system, where two ‘paths’ can be assigned to the two basis states, he showed that uncertainty and complementarity would mutually imply each other. Unfortunately he was unable to explicitly connect his results with the original proposal [7] and Rempe’s scheme, respectively. It was already in 1994 when Storey *et al.* [14,15] argued that the wave-particle duality should always be enforced by some momentum transfer as might be expected from the standard uncertainty relation. We intend to clarify this controversy by referring to generalized uncertainty relations for composite systems defined in a finite-dimensional state space.

## 2 Generalized uncertainty relations

Let us first state the general form of uncertainty relations: for a given state operator  $\hat{\rho}$  and two observables  $\hat{A}_j$ ,  $j = 1, 2$ , with  $\langle \hat{A}_j \rangle = \text{Tr}(\hat{\rho} \hat{A}_j)$  denoting the respective expectation values, we define  $\delta \hat{A}_j = \hat{A}_j - \langle \hat{A}_j \rangle \hat{1}$ , where  $\hat{1}$  is the unit operator. Then, with the commutator

$$[\delta \hat{A}_1, \delta \hat{A}_2]_- = [\hat{A}_1, \hat{A}_2]_- = 2i \hat{B}, \quad (1)$$

the generalized uncertainty relations [16] read

$$(\Delta A_1)^2 (\Delta A_2)^2 \geq |\langle \hat{B} \rangle|^2 + \chi_{A_1 A_2}^2, \quad (2)$$

i.e. the product of the variances  $(\Delta A_j)^2 = \langle (\delta \hat{A}_j)^2 \rangle$  is bounded from below by the sum of two terms: the expectation value of  $\hat{B}$  and the symmetrized covariance

$$\chi_{A_1 A_2} = \frac{1}{2} \langle \delta \hat{A}_1 \delta \hat{A}_2 + \delta \hat{A}_2 \delta \hat{A}_1 \rangle. \quad (3)$$

The first term on the right hand side in (2) restricts the (ensemble-) measurement outcomes with respect to two non-commuting observables (for which  $\hat{B} \neq 0$ ). The second term accounts for the influence of correlations: it has a classical analog [16] and may be unequal zero even for two commuting observables. For the two canonically conjugate operators  $\hat{A}_1 = \hat{x}$  and  $\hat{A}_2 = \hat{p}_x$ , the commutator  $\hat{B}$  is proportional to the unit operator  $\hat{1}$ , so that [17]

$$(\Delta x)^2 (\Delta p_x)^2 \geq \frac{1}{4} \hbar^2 + \frac{1}{4} \langle \psi | [\delta \hat{x}, \delta \hat{p}_x]_+ | \psi \rangle^2. \quad (4)$$

The inequality refers to one given initial state, not to a sequential or even ‘simultaneous’ measurement of  $\hat{x}$  and  $\hat{p}_x$  on the same individual system. The last term in equation (4) is usually discarded as an explicitly state-dependent correction. Note, however, that the first (state-independent) term is not generic:

such a term cannot occur in any finite-dimensional Hilbert-space as the commutator must be traceless. But explicit state-dependence renders measurement-induced random kicks as the origin of quantum uncertainty a much less convincing concept.

### 3 Which-path detection model

It has been stated that at least the ‘historical’ which-path detectors in a double-slit configuration could be ‘explained’ in terms of the canonical relations (4). We doubt that: in Feynman’s light-scattering arrangement [6] a light beam is supposed to interact with the electrons after they have passed through the double-slits to reveal their paths. Though the momentum ‘kicks’ by the photons might explain why the interference patterns on the screen are washed out [14,15], but it remains open why this continuous random perturbation  $\Delta p_x$  should produce exactly 2(!) alternative patterns (the sub-ensembles originating from slit 1 as if slit 2 were closed and from slit 2 as if slit 1 were closed). The photon momentum appears to play a role, though, when Feynman proposes to reduce the perturbation by reducing the photon momentum, thus increasing the wavelength. The ‘which-path’ detector ceases to work when the slits (the two paths) can no longer be resolved, and the interference patterns reappear. However, rather than being a result of reduced ‘random kicks’, this should be seen as a logical consequence of the fact that in this limit the ‘which-path’ detector is unable to distinguish the two paths, so that the necessary correlation cannot build up (see below). Similar arguments should apply to Einstein’s recoiling slit [4], as well as to early experimental realizations [18].

Interference may result when one final state of a quantum object can be reached in at least two different ways, e.g. from two different initial states. Typical experimental settings involve pre-selected paths. Even though spatial- or momentum-coordinates form a continuous set, the experimental design for a ‘which-path’ detection typically reduces that space to discrete (usually two) alternatives, just as the electronic levels of an atom, say, can be se-

lected to simulate a two-level system (pseudo-spin) [19]. Such an effective two-level system is most conveniently described in terms of the Pauli operators  $\hat{\sigma}_j$ ,  $j = x, y, z$  with  $(\hat{\sigma}_j)^2 = \hat{1}$  and e.g.  $[\hat{\sigma}_x, \hat{\sigma}_y]_- = 2i \hat{\sigma}_z$ . As a consequence the state  $\hat{\rho}$  can be specified by the Bloch vector  $\sigma_j = \langle \hat{\sigma}_j \rangle$  and its variance by  $0 \leq (\Delta\sigma_j)^2 = 1 - (\sigma_j)^2 \leq 1$ . Finally, using  $\chi_{\sigma_x\sigma_y} = -\sigma_x\sigma_y$ , equation (2) now implies

$$(\Delta\sigma_x)^2 (\Delta\sigma_y)^2 \geq (\sigma_z)^2 + (\sigma_x)^2 (\sigma_y)^2. \quad (5)$$

Further inequalities are obtained by permuting the indices  $x, y, z$ . This inequality constitutes a relation between first and second moments; there is no state-independent term. Nevertheless, we see that perfect knowledge of  $\hat{\sigma}_z$ , say,  $(\sigma_z = \pm 1)$  leads necessarily to complete ignorance about  $\hat{\sigma}_x$  and  $\hat{\sigma}_y$  ( $\Delta\sigma_x = \Delta\sigma_y = 1$ , i.e.  $\sigma_x = \sigma_y = 0$ ). Obviously, these three observables are pairwise complementary, without being conjugate like the canonical variables in (4).

In the following we will extend these considerations to composite systems (composed of distinguishable subsystems). While the local observables referring to different subsystems are always commutative, they both turn out to be complementary to quantum correlation (the covariance term in (2)). This conclusion can be reached in various ways; here we will show that it can be derived from an appropriate uncertainty relation.

Interference between a double-slit results from the superposition of two different paths (modes). The two paths will be identified as the eigenstates of  $\hat{\sigma}_z$ ,  $\hat{\sigma}_z |\pm z\rangle = \pm |\pm z\rangle$ . Interference then requires the preparation of superposition states like  $A|+z\rangle + B|-z\rangle$ . In the concrete double-slit experiment,  $A, B$  will depend on the spatial coordinates on the screen. Here we restrict ourselves to the form

$$|\varphi\rangle = \frac{1}{\sqrt{2}} \left( |+z\rangle + e^{i\varphi} |-z\rangle \right). \quad (6)$$

The states  $|\varphi = 0\rangle = |+x\rangle$ ,  $|\varphi = \pi\rangle = |-x\rangle$  are formal eigenstates of  $\hat{\sigma}_x$ . At any point of the screen we measure  $|\varphi = 0\rangle$ , i.e. the corresponding probability

$P_0(\varphi) = |\langle 0|\varphi\rangle|^2 = \frac{1}{2}(1 + \sigma_x) = \frac{1}{2}(1 + \cos \varphi)$ , which constitutes our idealized interference fringes with fringe visibility

$$\mathcal{V} = \frac{P_0(\varphi)_{\max} - P_0(\varphi)_{\min}}{P_0(\varphi)_{\max} + P_0(\varphi)_{\min}} = 1. \quad (7)$$

The quantitative measure,  $\mathcal{D} = 0$ , for the  $|\pm z\rangle$ -distinguishability is then obtained from  $\mathcal{D}^2 + \mathcal{V}^2 = 1$  [19].

While the interference pattern becomes visible only as an ensemble result, it is, nevertheless, a single-particle property. Ideal ‘which-path’ detection therefore requires a single-particle sensitivity. By such a ‘which-path’ detection scheme, a physical label is introduced to mark those ‘paths’ i.e. to make them ‘distinguishable’. For this purpose we introduce a second two-level system (subsystem 2, e.g. an internal atomic two-level system or the polarisation states of a photon), as proposed in [7] (see also [20]): the ‘which-path’ detection requires to build up a strict correlation or anti-correlation between the paths,  $|\pm z\rangle^{(1)}$ , and the marker states,  $|\pm z\rangle^{(2)}$ , e.g.  $|+z\rangle^{(1)} \otimes |+z\rangle^{(2)}$ ,  $|-z\rangle^{(1)} \otimes |-z\rangle^{(2)}$ . Because of the linearity of quantum mechanics the local coherent state of subsystem 1 evolves into

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}} (|+z, +z\rangle + |-z, -z\rangle), \quad (8)$$

where the first index refers to subsystem 1 (path index), and the second one to subsystem 2 (marker). This state can formally be obtained by means of a quantum-controlled NOT operation [21]: after subsystem 2 has been prepared in a standard state, say,  $|-z\rangle^{(2)}$ , we apply the unitary transformation  $|-z\rangle^{(2)} \rightarrow |+z\rangle^{(2)}$  if the state of subsystem 1 is  $|+z\rangle^{(1)}$ , no change otherwise (transition table:  $|+z, +z\rangle \rightarrow |+z, -z\rangle$ ,  $|+z, -z\rangle \rightarrow |+z, +z\rangle$ ,  $|-z, +z\rangle \rightarrow |-z, +z\rangle$ ,  $|-z, -z\rangle \rightarrow |-z, -z\rangle$ ).

## 4 Uncertainty versus complementarity

The total system can be described in terms of the two-particle operators,  $\hat{K}_{jk} = \hat{\sigma}_j^{(1)} \otimes \hat{\sigma}_k^{(2)}$ ,  $j, k = x, y, z$ , the single-particle operators,  $\hat{K}_{j0} = \hat{\sigma}_j^{(1)} \otimes \hat{1}^{(2)}$ ,  $\hat{K}_{0k} = \hat{1}^{(1)} \otimes \hat{\sigma}_k^{(2)}$ , and  $\hat{K}_{00} = \hat{1} = \hat{1}^{(1)} \otimes \hat{1}^{(2)}$  [21]. The incompatibility between the single- and the two-particle operators can be specified by

$$\begin{aligned} [\hat{K}_{jj}, \hat{K}_{0k}]_- &= \hat{\sigma}_j^{(1)} \otimes [\hat{\sigma}_j^{(2)}, \hat{\sigma}_k^{(2)}]_- \neq 0 \quad \text{for } j \neq k \\ [\hat{K}_{jj}, \hat{K}_{l0}]_- &\neq 0 \quad \text{for } j \neq l. \end{aligned}$$

The single-particle operators acting on different subsystems obviously commute. Their covariance, equation (3), is [22]

$$\chi_{\sigma_j^{(1)} \sigma_k^{(2)}} = K_{jk} - \sigma_j^{(1)} \sigma_k^{(2)} = M_{jk}, \quad (9)$$

where  $K_{jk} = \langle \hat{K}_{jk} \rangle$  describes two-particle correlations. One easily convinces oneself that in general  $0 \leq |M_{jk}| \leq 1$ , while  $M_{jk} = 0$  for product states [23]. Accordingly, our new inter-subsystem uncertainty relations are given by

$$\left(\Delta \sigma_j^{(1)}\right)^2 \left(\Delta \sigma_k^{(2)}\right)^2 \geq |M_{jk}|^2, \quad j, k = x, y, z. \quad (10)$$

In this case the uncertainty relations in the conventional form, i.e. discarding the covariance term, would be absolutely insufficient. This inequality is used here to assess quantum mechanical properties even though its form would apply also to any pair of classical statistical variables  $\sigma_j^{(1)}, \sigma_k^{(2)}$ . For pure states,  $M_{jk}$  implies non-local quantum correlations, and these quantum effects can still be manifest in equation (10): for the case  $|M_{jk}| = 1$ , which means strict (anti)-correlation, one obtains a maximum ignorance of  $\hat{\sigma}_j^{(1)}$  and  $\hat{\sigma}_k^{(2)}$  ( $\Delta \sigma_j^{(1)} = \Delta \sigma_k^{(2)} = 0$ ). On the other hand, the perfect knowledge about  $\hat{\sigma}_j^{(1)}$  and  $\hat{\sigma}_k^{(2)}$  leads to maximum uncertainty in the correlation,  $|M_{jk}| = 0$ .

The state (8) (with distinguishability  $\mathcal{D} = 1$  for the ‘which-path’ information) is a joint eigenfunction of the commuting set  $\hat{K}_{jj}$ ,  $j = x, y, z$  and has

$|M_{ij}| = \delta_{ij}$ . It thus shows zero fringe visibility,  $\mathcal{V} = 0$  ( $P_0(\varphi) = \text{const.}$ ). The preparation of the entangled state implies that each subsystem will be in a non-pure state. This fact may seem to justify an interpretation in terms of a ‘mixture’ resulting from some randomization (e.g. due to ‘photon kicks’, see [6,14,15,24]). But this picture becomes inconsistent as we are able to completely remove the alleged randomization by post-selection (‘quantum erasure’). To see this we observe that equation (8) can be rewritten as

$$|\psi(1, 2)\rangle = \frac{1}{\sqrt{2}} (|+x, +x\rangle + |-x, -x\rangle).$$

If we now sort out  $|+x\rangle^{(2)}$  or  $|-x\rangle^{(2)}$  - and only then - we recover the respective interference patterns, as proposed by Scully *et al.* [7]. This result is in accord with the uncertainty relation (10), as this sorting out requires a measurement and after having obtained  $\sigma_x^{(1)} = \sigma_x^{(2)} = +1$  or  $-1$ , one of which is post-selected, we obtain  $\Delta\sigma_x^{(1)} = \Delta\sigma_x^{(2)} = 0$  and thus  $M_{xx} = 0$ , indicating a product state with local coherence (cf. equation (6)).

Other marker states would also work. As an example we consider a two-spin system (subsystem 2,3) and choose as the initial state

$$|\psi(2, 3)\rangle = \frac{1}{\sqrt{2}} (|+z, +z\rangle + |-z, -z\rangle).$$

If the coupling is the same as before (CNOT between subsystem 1 and 2), we have

$$|\psi(1, 2, 3)\rangle = \frac{1}{2} \left( | +z \rangle^{(1)} \otimes (| -z, +z \rangle + | +z, -z \rangle) + | -z \rangle^{(1)} \otimes (| +z, +z \rangle + | -z, -z \rangle) \right),$$

where the two Bell states in subspace (2,3) now play the role of an effective two-state subsystem. Note that the subsystems 2 and 3 are both in a ‘mixed’ state for each alternative ‘path’, carrying no local information! Such a situation has been investigated by Kwiat *et al* [9].



## 5 Summary and discussion

We have tried to convince the reader that - as we include more general scenarios referring to composite systems - (1) complementarity and uncertainty relations can still strictly be related, (2) quasi-classical explanations for the origin of uncertainty (complementarity) in terms of measurement-induced “random kicks” become untenable. Recent ‘which-path’ measurement schemes challenge the strict interrelation between uncertainty and complementarity: it is argued that entanglement may lead to complementarity without uncertainty relation. In fact, single-particle coherence ( $M_{jk} = 0$ ) and two-particle coherence derived from entanglement ( $M_{jk} \neq 0$ ) are complementary [25].

Here we noted that generalized uncertainty relations for composite quantum systems will, in general, depend not only on the commutator of the two observables considered, but also on mutual correlations, in particular, on entanglement. It thus follows that the equivalence between the set of uncertainty relations and Bohr’s complementarity holds in a much broader sense than believed up to now. Equation (10) is the pertinent uncertainty relation underlying binary ‘which-path’ measurements, not the intra-subsystem relation (4). The former is but one out of a large class of hitherto unexplored inter-subsystem uncertainty relations, which can be generalized to any subsystem size and for which entanglement should turn out to be a natural ingredient. These inequalities between expectation values would be confirmed by appropriate ensemble measurements. But their interpretation in terms of ‘classical’ measurement disturbances - already of limited qualitative value in simple cases - has to be abandoned in the general case:

For a composite quantum system in a pure-state, entanglement implies that each subsystem is in a non-pure state, which would justify an interpretation in terms of a ‘mixture’ resulting from some randomization. However, even though the maximum entangled state has lost its local interference properties, it exhibits, when correlated with the ‘which-path’ measurement outcome, the same pattern as generated by either slit, without any additional randomness.

Rather than talking of local ‘random kicks’, we may say that each measure-

ment ‘projects’ the composite system in a correlated fashion into one of the states  $|\pm z\rangle^{(1)}$  and thus subsystem 1 into one of the alternate paths.

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